

जून 2009

D.G. Sem-2nd  
paper-VI, unit-1st  
Complex integration  
Dr. G. D. Singh. 8-6-2021

(1)

21 रवि Note: The above theorem can also be asked in this way. If  $f(z)$  is analytic in a simply connected region  $D$  of the complex plane, show that there exists a function  $F(z)$  analytic in  $D$ , and such that  $F'(z) = f(z)$  for  $z$  in  $D$ .

[IAS 70]

Theorem 5 Taylor's theorem :->

If a function  $f(z)$  is analytic within a circle  $C$  with its centre  $z = a$  and radius  $R$ ,

22 सोम then at every point  $z$  inside  $C$

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (z-a)^n$$

Or

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n \text{ where } a_n = \frac{f^{(n)}(a)}{n!}$$

I.A.S.

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Let  $f(z)$  be analytic within a circle  $C$  whose equation is  $|t-a| = R$ .

Let  $z$  be any point within  $C$  such that  $|z-a| = r < R$

By Cauchy's integral formula

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(t)}{(t-z)} dt$$

$$= \frac{1}{2\pi i} \int_C \frac{f(t) dt}{t-a-z+a}$$

$$= \frac{1}{2\pi i} \int_C \frac{f(t) dt}{(t-a)-(z-a)}$$

$$= \frac{1}{2\pi i} \int_C \frac{f(t)}{(t-a) \left\{ 1 - \frac{z-a}{t-a} \right\}} dt$$

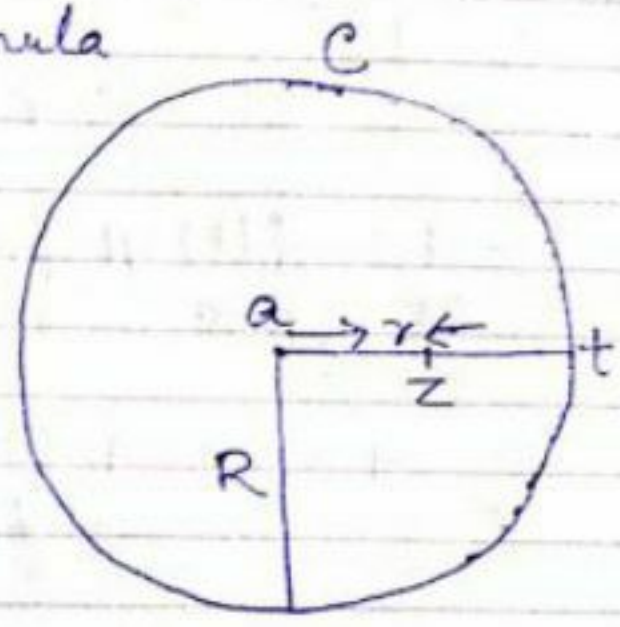
$$= \frac{1}{2\pi i} \int_C \frac{f(t)}{t-a} \left\{ 1 - \frac{z-a}{t-a} \right\}^{-1} dt \quad \text{--- (1)}$$

Applying the formula

$$\frac{1}{1-b} = (1-b)^{-1} = 1 + b + b^2 + b^3 + \dots + b^{n-1} + b^n + b^{n+1} + b^{n+2} + \dots$$

$$= 1 + b + b^2 + b^3 + \dots + b^{n+1} (1 + b + b^2 + b^3 + \dots)$$

$$= 1 + b + b^2 + b^3 + \dots + b^{n+1} (1-b)^{-1}$$



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So we have from (1)

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$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(t)}{(t-a)} \left[ 1 + \frac{z-a}{t-a} + \left(\frac{z-a}{t-a}\right)^2 + \dots + \left(\frac{z-a}{t-a}\right)^n + \left(\frac{z-a}{t-a}\right)^{n+1} \left(1 - \frac{z-a}{t-a}\right)^d \right] dt$$

$$= \frac{1}{2\pi i} \int_C \frac{f(t)}{t-a} dt + (z-a) \frac{1}{2\pi i} \int_C \frac{f(t)}{(t-a)^2} dt + (z-a)^2 \frac{1}{2\pi i} \int_C \frac{f(t)}{(t-a)^3} dt$$

$$+ \dots + \frac{(z-a)^{n+1}}{2\pi i} \int_C \frac{f(t) dt}{(t-a)^{n+1} \left(1 - \frac{z-a}{t-a}\right)}$$

$$= \frac{1}{2\pi i} \int_C \frac{f(t)}{t-a} dt + \frac{z-a}{2\pi i} \int_C \frac{f(t)}{(t-a)^2} dt + \frac{(z-a)^2}{2\pi i} \int_C \frac{f(t)}{(t-a)^3} dt$$

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$$+ \dots + \frac{(z-a)^n}{2\pi i} \int_C \frac{f(t) dt}{(t-a)^{n+1}}$$

$$+ \frac{(z-a)^{n+1}}{2\pi i} \int_C \frac{f(t)}{(t-a)^{n+2} \left(\frac{t-a-z+a}{t-a}\right)}$$

$$\text{Or } f(z) = \frac{1}{2\pi i} \int_C \frac{f(t)}{t-a} dt + \frac{z-a}{2\pi i} \int_C \frac{f(t)}{(t-a)^2} dt + \frac{(z-a)^2}{2\pi i} \int_C \frac{f(t)}{(t-a)^3} dt + \dots + \frac{(z-a)^{n+1}}{2\pi i} \int_C \frac{f(t)}{(t-a)^{n+2-1} (t-z)} dt \quad (2)$$

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We know

$$f^{(h)}(a) = \frac{h!}{2\pi i} \int_C \frac{f(z) dz}{(z-a)^{h+1}}$$

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$$\therefore \frac{1}{2\pi i} \int_C \frac{f(z) dz}{(z-a)^{h+1}} = \frac{f^{(h)}(a)}{h!} \quad \text{--- (3)}$$

From (2)

$$f(z) = f(a) + \frac{z-a}{1!} f'(a) + \frac{(z-a)^2}{2!} f''(a) + \dots + \frac{(z-a)^h}{h!} f^{(h)}(a) + U_{h+1} \quad \text{--- (4)}$$

where  $U_{h+1} = \frac{(z-a)^{h+1}}{2\pi i} \int_C \frac{f(t) dt}{(t-z)(t-a)^{h+1}}$

$$|U_{h+1}| \leq \frac{|z-a|^{h+1}}{2\pi} \int_C \frac{|f(t)| |dt|}{(1+|t|-|z|) |t-a|^{h+1}}$$

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$$\leq \frac{M}{2\pi} \left(\frac{r}{R}\right)^{h+1} \frac{1}{R-r} 2\pi R$$

where  $M = \max |f(t)|$  on  $C$

$$\text{or } |U_{h+1}| \leq M \left(\frac{r}{R}\right)^{h+1} \frac{1}{1 - \frac{r}{R}} \rightarrow 0 \text{ as } h \rightarrow \infty$$

$$\left[ \because \lim_{h \rightarrow \infty} \left(\frac{r}{R}\right)^{h+1} = 0 \text{ as } \frac{r}{R} < 1 \right]$$

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(5)

29 सोम  $\therefore \lim_{h \rightarrow \infty} U_{n+1} = 0$  making  $n \rightarrow \infty$  in (4)

We get

$$f(z) = \lim_{h \rightarrow \infty} \left[ f(a) + \frac{z-a}{1!} f'(a) + \frac{(z-a)^2}{2!} f''(a) + \dots + \frac{(z-a)^n}{n!} f^{(n)}(a) \right]$$

$$= \sum_{h=0}^{\infty} \frac{(z-a)^h}{h!} f^{(h)}(a) = \sum_{h=0}^{\infty} (z-a)^h \frac{f^{(h)}(a)}{h!}$$

$$= \sum_{n=0}^{\infty} a_n (z-a)^n \quad \text{--- (5)}$$

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where  $a_n = \frac{f^{(n)}(a)}{n!}$

found

**Deduction:**

Since  $z$  is a pt within the circle  $|z-a| = R$  such that  $|z-a| = r < R$  so that we can take  $z = a+h$  or  $h = z-a$

from (5)

$$f(z) = \sum_{h=0}^{\infty} \frac{h^h}{h!} f^{(h)}(a)$$

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$$= f(a) + h f'(a) + \frac{h^2}{2} f''(a) + \dots$$

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this is alternative form of Taylor's series.

(ii) if we write  $a=0$  in (5)

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} z^n$$

this is known as Maclaurin's series.